Mechanics: Oscillations

FIZIKA SPhO Training

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Contents

1 Notes

1.1 Simple Harmonic Motion

1.1.1 Characteristic Equation

Simple harmonic motion (SHM) is the motion of a particle in which

1. its acceleration is directly proportional to its displacement from a fixed equilibrium position

2. it is always directed towards that equilibrium position.

Equivalently, if $x(t)$ denotes the displacement at time t, the definity equation of motion is:

$$a = \frac{d^2x}{dt^2} = -\omega^2 x \tag{1}$$

where the angular frequency $\omega = 2\pi f = \frac{2\pi}{T}$.

By solving the differential equation, the general equation of motion is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{2}$$

for some constants C_1 and C_2, depending on the initial conditions.

Equivalently, using the R-formula,

$$x(t) = x_0 \cos(\omega t + \varphi) \tag{3}$$

where x_0 is the amplitude, and φ is the phase.

There are other useful equations that stem from the characteristic equation:

- Maximum Values: $v_{\max} = A\,\omega$, $a_{\max} = A\,\omega^2$

- Total Energy: $E = \frac{1}{2}k\,A^2 = \frac{1}{2}m\,\omega^2 A^2$

- Kinetic Energy: $K = \frac{1}{2}m\,v^2 = \frac{1}{2}m\,\omega^2(A^2 - x^2)$

- Potential Energy: $U = \frac{1}{2}k\,x^2 = \frac{1}{2}m\,\omega^2\,x^2$

Example 1.1. Try to derive the functions $v(x)$ and $a(v)$ and then plot them.
Eliminate the time variable with $\sin^2\theta + \cos^2\theta = 1$:

$$v^2 = A^2\omega^2\left(1 - \cos^2\theta\right) = \omega^2(A^2 - x^2) \quad \implies \quad \boxed{v(x) = \pm\omega\sqrt{A^2 - x^2}}.$$

Use the change-of-variables trick $a = v\,dv/dx$. From $v^2 = \omega^2(A^2 - x^2)$:

$$\frac{dv}{dx} = \frac{-\omega^2 x}{v} \quad \implies \quad a = v\frac{dv}{dx} = -\omega^2 x.$$

Solving the earlier relation for x in terms of v, $x = \pm\sqrt{A^2 - \frac{v^2}{\omega^2}}$, gives the *multi-valued* (upper $+$ or lower $-$ branch) expression

$$\boxed{a(v) = \mp\omega^2\sqrt{A^2 - \frac{v^2}{\omega^2}}}.$$

Features of Graphs:

- $a(x)$ **straight line:** slope $-\omega^2$; endpoints $(x = \pm A,\ a = \mp A\omega^2)$.

- $v(x)$ **ellipse:** intercepts at $(x = \pm A,\ v = 0)$ and $(x = 0,\ v = \pm A\omega)$.

- $a(v)$ **ellipse:** intercepts at $(v = \pm A\omega,\ a = 0)$ and $(v = 0,\ a = \pm A\omega^2)$.

1.1.2 Physical Problems

To solve such problems, we should always aim to write the equations of motion using **N2L**.

Example 1.2. A rod of mass m is put on top of two oppositely rotating cylinders with coefficient of friction μ, separated by a distance $2L$. The centers of the cylinders are fixed. If the rod is displaced from equilibrium, describe the subsequent motion of the rod, and calculate a property of the motion.
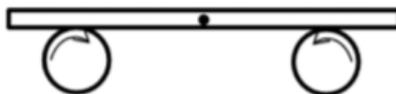
Denote the normal reactions at the left and right supports by N_1 and N_2, respectively. To begin, suppose the rod was displaced by a distance x from equilibrium to right:

$$N_1 + N_2 = mg \quad \text{(vertical force balance)},$$
$$N_1(L + x) = N_2(L - x) \quad \text{(torque balance about the centre of mass)}.$$

Solving,
$$N_2 = \frac{mg\,(L + x)}{2L}, \qquad N_1 = \frac{mg\,(L - x)}{2L}.$$

The magnitude of the kinetic-friction force at each support is μN_i.

$$F_{f,1} = +\mu N_1 \quad \text{(to the right)}, \qquad F_{f,2} = -\mu N_2 \quad \text{(to the left)}.$$

Hence the net horizontal force on the rod is

$$F = \mu N_1 - \mu N_2 = \mu m g \left(\frac{L - x}{2L} - \frac{L + x}{2L} \right) = -\frac{\mu m g}{L}\, x.$$

$$m\ddot{x} = -\frac{\mu m g}{L}\, x \quad \Longrightarrow \quad \ddot{x} + \underbrace{\left(\frac{\mu g}{L} \right)}_{\omega^2} x = 0.$$

This is the characteristic differential equation of *simple harmonic motion* with angular frequency

$$\boxed{\omega = \sqrt{\frac{\mu g}{L}}}$$

Example 1.3. To tackle problems where the SHM takes place about a shifted equilibrium, we adopt a general method, illustrated with the following example: A mass m hangs from a vertical massless spring of stiffness k with zero natural length. Let y be the height from the ceiling to the mass attached to the spring.

1. **Find the static equilibrium:**

$$k\, y_{\text{eq}} = m\, g \quad \Longrightarrow \quad y_{\text{eq}} = \frac{m\, g}{k}.$$

2. **Define a new variable, displacement about equilibrium:**

$$\delta y(t) = y(t) - y_{\text{eq}},$$

so that $\delta y = 0$ at the static equilibrium.

3. **Write the new equation of motion:**

$$m\ddot{y} = mg - ky \quad \Rightarrow \quad m\left(\ddot{\delta y} + \ddot{y_{eq}}\right) = mg - k\left(\delta y + y_{eq}\right)$$

Since y_{eq} is constant, $\ddot{y_{eq}} = 0$. Substituting in y_{eq}, we get

$$m\,\ddot{\delta y} + k\,\delta y = 0.$$

4. **Apply SHM solution:**

$$\ddot{\delta y} + \omega^2\,\delta y = 0, \quad \omega = \sqrt{\frac{k}{m}},$$

It turns out the frequency stays the same.

Example 1.4. A solid cube of side length b with density ρ_c is floating in a certain liquid with density ρ_l. It is pushed down slightly then released. You may assume the liquid level stays fixed (there is no displacement of liquid). Describe the subsequent motion in detail.

Let's follow the same framework in the previous example. Let h_0 be the submerged depth of the cube at rest. Buoyant force = weight:

$$\rho_l g(b^2 h_0) \;=\; \rho_c g(b^3) \quad \Longrightarrow \quad h_0 = \frac{\rho_c}{\rho_l}\,b.$$

Define a small vertical displacement

$$\delta y(t) = h(t) - h_0,$$

measured *downward*. Thus $\delta y > 0$ means the cube is pushed farther into the liquid, and $\delta y = 0$ corresponds to equilibrium.

The submerged volume becomes $V_{\text{sub}} = b^2(h_0 + \delta y)$; hence the buoyant force is

$$F_B = \rho_l g\,b^2(h_0 + \delta y).$$

The cube's weight is unchanged, $mg = \rho_c g b^3$. The net upward force is

$$F_{\text{net}} = F_B - mg = \rho_l g b^2(h_0 + \delta y) - \rho_c g b^3.$$

Using the equilibrium relation $\rho_l g b^2 h_0 = \rho_c g b^3$ cancels the constants:

$$F_{\text{net}} = -\,\rho_l g\,b^2\,\delta y.$$

Newton's 2nd Law with mass $m = \rho_c b^3$:

$$\rho_c b^3\,\ddot{\delta y} = -\,\rho_l g\,b^2\,\delta y \quad \Longrightarrow \quad \ddot{\delta y} + \frac{\rho_l g}{\rho_c b}\,\delta y = 0.$$

The standard SHM form $\ddot{\delta y} + \omega^2 \delta y = 0$ identifies

$$\boxed{\omega = \sqrt{\frac{\rho_l g}{\rho_c b}}}$$

Hence the cube oscillates vertically about its equilibrium depth with angular frequency ω, provided $\rho_c < \rho_l$ (so the cube floats).

1.1.3 Small Oscillations

A common trick very often used to tackle small oscillations is to use the binomial expansion and take low order terms:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots .$$

For $|x| \ll 1$, higher-order terms become negligible, so to first order

$$\boxed{(1+x)^n \approx 1 + nx.}$$

By taking first order terms, we essentially "linearize" many forces, hence many situations become simple harmonic. However, if your answer does not make physical sense, you might have to consider a 2nd order or higher term.

Example 1.5 (SPOT TST 2022). Suppose that two point masses m_1 and m_2 separated by a distance r interact via the potential

$$U = \frac{\kappa m_1 m_2}{r^n}, \qquad \kappa > 0,\ n \in \mathbb{Z}_{>0}.$$

State, with brief reasons, whether the corresponding force is attractive or repulsive.
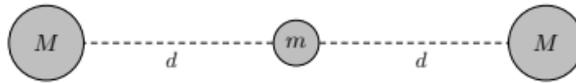
Now, two identical fixed masses M are a distance $2d$ apart on a straight line. A third mass m is free to move along the line segment between them (so the separations are $d \pm x$ with $|x| \ll d$). Determine, in terms of the symbols introduced, the angular frequency ω of the small oscillations of m about its equilibrium position.

To solve the first part of the question, note that the force corresponding to the given potential is

$$F(r) = -\frac{dU}{dr} = -\frac{d}{dr}\left(\frac{\kappa m_1 m_2}{r^n}\right) = +\frac{n\kappa m_1 m_2}{r^{n+1}} > 0,$$

directed *outwards* along the line joining the masses (it increases the separation). Hence the interaction is **repulsive**.

When the mobile mass m is displaced a small distance x to the right of the midpoint,

$$U(x) = \frac{\kappa m M}{(d+x)^n} + \frac{\kappa m M}{(d-x)^n}.$$

Factor out d^{-n} and expand each term with the binomial series:

$$(d \pm x)^{-n} = d^{-n}\left(1 \pm \frac{x}{d}\right)^{-n} \simeq d^{-n}\left[1 \mp n\frac{x}{d} + \tfrac{1}{2}n(n+1)\left(\frac{x}{d}\right)^2\right].$$

Adding the two expansions cancels the linear term, giving

$$U(x) = 2\kappa m M d^{-n}\left[1 + \tfrac{1}{2}n(n+1)(\tfrac{x}{d})^2\right] = U_0 + \kappa m M \frac{n(n+1)}{d^{n+2}} x^2.$$

Comparing $U(x) = U_0 + \frac{1}{2} k_{\text{eff}} x^2$ with the last line,

$$k_{\text{eff}} = 2\kappa M n(n+1)\, d^{-(n+2)}.$$

For a particle of mass m in a quadratic potential, $\frac{1}{2} m \omega^2 = \frac{1}{2} k_{\text{eff}}$. Hence

$$\boxed{\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2\kappa M\, n(n+1)}{d^{n+2}}}}.$$

The motion is therefore simple harmonic about the midpoint provided the displacement is small compared with d. We notice here that we had to make a 2nd order expansion - had we expanded to 1st order, the answer would not have made physical sense. Try it yourself!

1.2   Energy in Oscillatory Systems

Any conservative mechanical system can be described by a potential energy function $U(q)$, where q is the generalized coordinate. Equilibrium positions q_0 satisfy

$$\left.\frac{\mathrm{d}U}{\mathrm{d}q}\right|_{q=q_0} = 0.$$

At these points, there is no net force, since $F = -\frac{\mathrm{d}U}{\mathrm{d}q}$

Additionally, note that even though an object is in equilibrium, it may be in unstable or stable equilibrium. This can be deduced from the second derivative of the potential energy. If $U''(q) < 0$, this indicates an unstable equilibrium. If $U''(q) > 0$, it indicates a stable equilibrium.

Taylor Expansion around Equilibrium

For small displacements $x = q - q_0$ about the stable equilibrium, expand $U(q)$ in a Taylor series:

$$U(q) = U(q_0) + \left.\frac{\mathrm{d}U}{\mathrm{d}q}\right|_{q_0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2 U}{\mathrm{d}q^2}\right|_{q_0} x^2 + \cdots .$$

Since the first derivative at equilibrium vanishes as it is equal to 0 at equilibrium, to second order:

$$U(q) \approx U(q_0) + \frac{1}{2} U''(q_0)\, x^2.$$

We may drop the constant $U(q_0)$ as it does not affect the dynamics. This was explained previously on page 6 (b).

1. Potential: $U(x) = \frac{1}{2}\, U''(q_0)\, x^2, \quad x = q - q_0.$

2. Force from potential: $F(x) = -\dfrac{\mathrm{d}U}{\mathrm{d}x} = -\, U''(q_0)\, x.$

3. Newton's 2nd law: $m\,\ddot{x} = F(x) = -\, U''(q_0)\, x \implies m\,\ddot{x} + U''(q_0)\, x = 0.$

4. Compare with SHM: $\ddot{x} + \omega^2 x = 0 \implies \omega^2 = \dfrac{U''(q_0)}{m}, \quad \boxed{\omega = \sqrt{\dfrac{U''(q_0)}{m}}}.$

Example 1.6. A particle of mass m moves in one dimension under the potential

$$U(x) = U_0\left(\frac{x^2}{a^2} - 1\right)^2,$$

where $U_0 > 0$ and $a > 0$ are constants.

(a) Find all equilibrium points x_0 by solving $U'(x_0) = 0$.

(b) Determine which of these are stable equilibria.

(c) Using a Taylor expansion of $U(x)$ about the stable equilibrium x_0, show that for small oscillations the motion is simple harmonic with angular frequency

$$\omega = \sqrt{\frac{U''(x_0)}{m}} .$$

(d) Compute $U''(x_0)$ at the stable point and hence write down the explicit expression for ω in terms of U_0, m and a.

This example is a direct application of the formulas above.

(a) We have

$$U(x) = U_0\left(\tfrac{x^2}{a^2} - 1\right)^2 = U_0\left(\tfrac{x^4}{a^4} - 2\,\tfrac{x^2}{a^2} + 1\right).$$

Compute

$$U'(x) = U_0\left(\tfrac{4x^3}{a^4} - \tfrac{4x}{a^2}\right) = \frac{4U_0\,x}{a^2}\left(\tfrac{x^2}{a^2} - 1\right).$$

Setting $U'(x) = 0$ gives

$$x = 0, \quad \text{or} \quad \frac{x^2}{a^2} = 1 \implies x = \pm a.$$

(b) Next,

$$U''(x) = U_0\left(\tfrac{12x^2}{a^4} - \tfrac{4}{a^2}\right).$$

Evaluate at each equilibrium:

$$U''(0) = -\frac{4U_0}{a^2} < 0 \quad \text{(unstable)}, \qquad U''(\pm a) = U_0\left(\tfrac{12a^2}{a^4} - \tfrac{4}{a^2}\right) = \frac{8U_0}{a^2} > 0 \quad \text{(stable)}.$$

(c) Around a *stable* equilibrium $x_0 = \pm a$, we write

$$U(x) \approx U(x_0) + \tfrac{1}{2}\,U''(x_0)\,(x - x_0)^2.$$

The effective potential near x_0 is that of a spring, $\tfrac{1}{2}\,m\,\omega^2\,(x - x_0)^2$, so

$$m\,\omega^2 = U''(x_0) \implies \omega = \sqrt{\frac{U''(x_0)}{m}}.$$

(d) Since $U''(x_0) = 8U_0/a^2$, we get

$$\boxed{\omega = \sqrt{\frac{8U_0}{m\,a^2}}.}$$

1.3 Damped/Driven Harmonic Oscillators

In reality, many systems are not ideal and are subject to external forces. Forces such as friction and air resistance that "dissipate" damping lead to **damped oscillations.** On the other hand, forces that "force" an oscillation lead to **driven oscillations.**

1.3.1   Solving the Damped Harmonic Oscillator

To solve for the equation of motion, we write N2L with the damping force as the net force.

Example 1.7. Consider a typical horizontal mass-spring system with mass m and spring constant k, but subject to a damping drag force $F_{drag} = -bv$ from air resistance, where $b > 0$. Find the general equation of motion.

By N2L,
$$m\ddot{x} = -kx - b\dot{x}$$

keeping in mind that the signs for both forces are negative, since both forces decrease the acceleration of the mass.

We may rearrange to get a 2nd order ODE:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

which can be solved using an exponential guess, $x(t) = Ae^{st}$. (Try it yourself!)

You should obtain

$$s = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

and thus, the general equation of motion is

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where s_1 and s_2 are the two solutions for s, and A_1 and A_2 are constants depending on the initial conditions.

We usually define the **damping coefficient,** γ and the **resonant/natural frequency,** ω_0:

$$\gamma := \frac{b}{2m} > 0 \tag{4}$$

$$\omega_0 := \sqrt{\frac{k}{m}} > 0 \tag{5}$$

and thus, s is usually written as

$$s = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6}$$

There are **three possible cases** of damping, depending on the term inside the square root.

1.3.2   Case 1: Underdamped Oscillations

Underdamped oscillations occur when $\gamma < \omega_0$. This means s is imaginary.
In this case, oscillations still exist. We can define the angular frequency as

$$\omega := \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{7}$$

Thus, $s = -\gamma \pm i\omega$, so we get

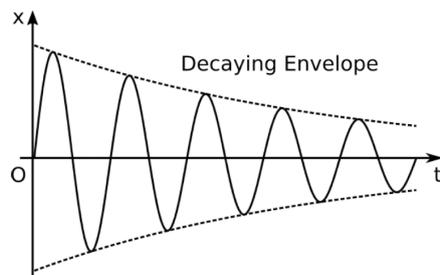$$x(t) = e^{-\gamma t}\left(A_1 \cos\left(\omega t\right) + A_2 \sin\left(\omega t\right)\right) \tag{8}$$

Figure 1: The term in front $e^{-\gamma t}$ is the equation of the decaying envelope. As $t \to \infty$, the amplitude rightfully decays to zero as all energy is lost.

1.3.3   Case 2: Critically Damped Oscillations

Critically damped oscillations occur when $\gamma = \omega_0$. This means that s is repeated.

In this case, oscillations don't exist. In particular, this is when the system returns to equilibrium in the shortest time possible. As $s = -\gamma, -\gamma$, we have

$$x(t) = (A_1 + A_2 t)\, e^{-\gamma t} \tag{9}$$
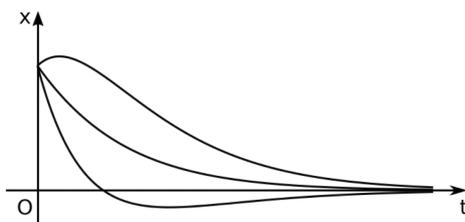
Figure 2: Critically damped systems decay to zero in the shortest time possible.

1.3.4   Case 3: Overdamped Oscillations

Overdamped oscillations occur when $\gamma > \omega_0$. This means that s is real and distinct.

In this case, oscillations don't exist. We simply have the general form

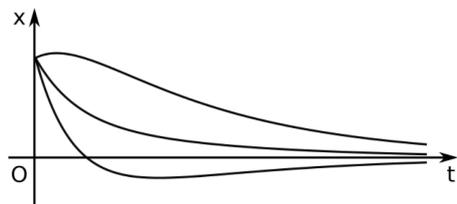$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{10}$$

Figure 3: Overdamped systems do not have oscillations and decay to zero.

Example 1.8. Without referring to the steps above, try deriving the equations of motion for the 3 different cases yourself.

1.3.5 Solving the Driven Harmonic Oscillator

Again, we write N2L, accounting for the driving force. The driving force is usually sinusoidal in nature.

Example 1.9. Consider the situation in Example 1.5, but there is now an additional applied force $F(t) = F_0 \cos(\omega t)$. Find the general equation of motion.

By N2L,

$$F_0 \cos(\omega t) - b\dot{x} - kx = m\ddot{x} \quad \Rightarrow \quad \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

To solve this non-homogeneous ODE, you would need to consider $x_{homogeneous}$ and $x_{particular}$, where $x_{homogeneous}$ is given by either Equation (8), (9) or (10) depending on the case of damping.

The sinusoidal form of the RHS means that we should guess

$$x_{particular} = A \cos(\omega t) + B \sin(\omega t)$$

By differentiating twice and substituting into the above equation, you should eventually obtain

$$A = \frac{\frac{F_0}{m}\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}, \quad B = \frac{2\frac{F_0}{m}\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}$$

The amplitude of the oscillation, by the R-formula, is thus given by

$$x_0 = \sqrt{A^2 + B^2} = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}}$$

Thus, the general equation of motion is

$$x(t) = x_{homogeneous} + x_{particular} = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}} \cos(\omega t + \varphi)$$

where all three cases of damping are accounted for by the values of s.

However, all three cases have something in common! They all **decay to zero eventually.** As such, we are usually not interested in the homogeneous solution, and we **only care about the particular solution.** The particular solution is also called the **steady-state solution.**

1.3.6 Resonance

Resonance is the state whereby there is **maximum energy transfer** from the driving force to the oscillating system.

When there is **no damping,** resonance occurs when $\omega = \omega_0$ (i.e. the driving frequency is equal to the natural frequency).

When there is **damping,** resonance occurs when the **amplitude is maximum.** From the form of x_0, we wish to minimise the term in the square root in the denominator, $\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2$, in order to maximise x_0. Thus,

$$\frac{d}{d\omega}\left(\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right) = 0 \quad \Rightarrow \quad \omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \tag{11}$$

At the resonant frequency ω_{res}, the maximum amplitude is

$$x_{0,max} = \frac{\frac{F_0}{m}}{2\gamma\sqrt{\omega_0^2 - \omega^2}} \tag{12}$$
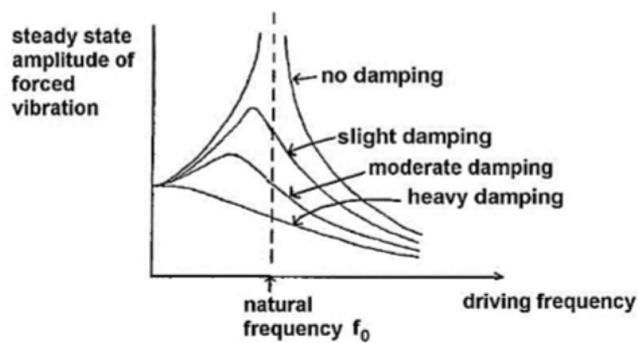
2   Problems

*Problems are arranged in roughly increasing difficulty.*

Problem 2.1 (SPhO 2018). (i) A horizontal platform vibrates with simple harmonic motion in the horizontal direction with a period of 2.0 s. A small object placed on the platform begins to slide when the amplitude of vibration reaches 0.4 m. Calculate the coefficient of static friction between the object and the platform. (ii) The platform now executes vertical simple harmonic motion with a period of 1.5 s. What is the maximum amplitude of the motion if the object were to be in contact with the plate throughout the motion?

Problem 2.2 (SPhO 2011). A large block, with a second block sitting on top, is connected to a spring and executes horizontal simple harmonic motion as it slides across a frictionless surface with an angular frequency ω. The coefficient of static friction between the two blocks is μ_s. Determine the maximum amplitude of oscillation if the upper block is not to slip.

Problem 2.3 (SPhO 2004). A T-shaped structure is formed from two uniform rods, A and B, each of length L and mass m, as shown below. A point mass m moves horizontally at right angles to rod A, strikes the end of rod B with an initial velocity V and sticks to it. (i) Find the position of the point on the T-shaped structure which remains stationary. (ii) Determine the angular velocity of the system immediately after the collision. (iii) Sketch the positions of the T-shaped structure in its subsequent motion. (iv) Determine the change in kinetic energy of the system as a result of the collision. (v) The point mass m is now removed, and the T-shaped structure is suspended from the free end of rod A and allowed to move freely in the plane of the "T". Determine the period of small oscillations.
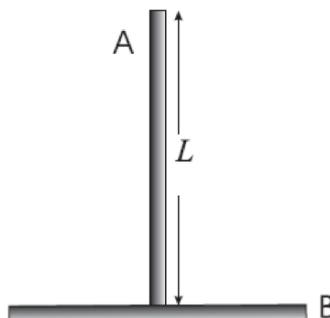

Problem 2.4 (SPhO 2005). A small thin disk of radius r and mass m is **attached rigidly** to the face of a second thin disk of radiu R and mass M, as shown below. The centre of the small disk is located at the edge of the large disk. The large disk is mounted at its centre on a frictionless horizontal axle. The assembly is rotated through an angle θ from its equilibrium position and released. (i) Find the speed of the centre of the small disk as it passes through the equilibrium position. (ii) Determine the period of small oscillations.
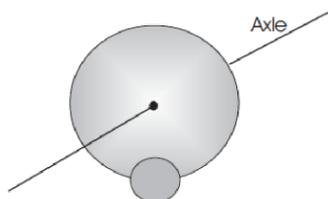

Problem 2.5 (SPhO 2002). A solid sphere with radius r and mass m rotates inside a **fixed** hollow sphere of radius R, as shown below. Calculate the period of small oscillations about the

equilibrium, assuming that the sphere rolls without slipping. The moment of inertia of the solid sphere about its central axis is $\frac{2}{5}mr^2$. (You'll need to know some rotation for this problem.)
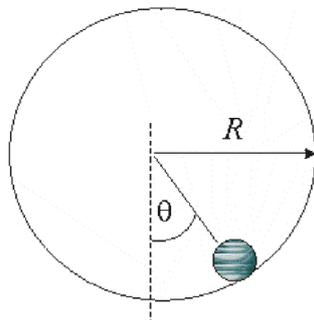

Problem 2.6. In the Cavendish experiment the two *small* balls have mass m each and are connected by a light rigid rod of length L. Two *large* balls have mass M each and their centres are also separated by the same distance L. The torsion constant of the suspension fibre is κ.

(a) *Large balls absent.* Find the period of the torsion pendulum.

(b) The large balls are placed a small distance away from the small balls. After a static equilibrium is reached the rod has rotated through a small angle θ and the centre–to–centre distance of each pair is r (see figure). Derive an expression for the gravitational constant G in terms of L, r, θ and the period T from part (a).

(c) If the small balls are perturbed through a *small* angle about the equilibrium in (b), determine whether the motion is simple harmonic when first-order changes in the gravitational force are included, and obtain the resulting period.
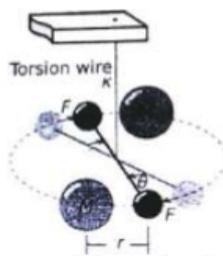

Problem 2.7. A solid cylinder of radius a and length ℓ, made of a lightweight material of uniform density d, is floating in seawater of density ρ. The gravitational acceleration is g. If the system is in equilibrium, write the expression relating the equilibrium tilt angle θ to the parameters a, ℓ, d, ρ, and g. The cylinder is then given a small downward perturbation from equilibrium, find the frequency of the resulting small oscillations. You may express your answer in terms of the equilibrium angle θ.
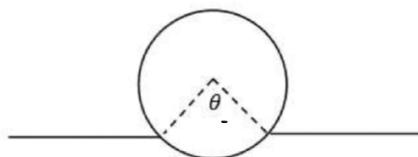

Problem 2.8 (SPhO 2014)**.** Problems involving springs often consider the springs to be massless. Of course, this is not true in reality. Here, we consider a spring of mass M, equilibrium length

L_0 and spring constant k. One end of the spring is fixed to a wall, and a mass m is attached at the other end of the spring. Neglecting gravity, determine the period of oscillations of this system. (Hint: Consider energy. How do you find the kinetic energy of the spring?)

Problem 2.9 (Cahn)**.** A particle of mass M is constrained to move on a frictionless horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane. The system is set-up so that the mass M moves in a circle of radius r, while the mass m remains still. Show that this motion is stable with respect to small changes in r, and find the angular frequency of small oscillations.
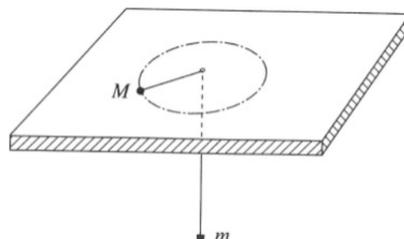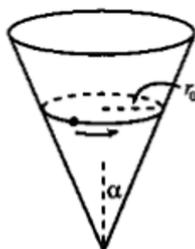

Problem 2.10. Two point masses of mass m are connected by a spring of spring constant k and relaxed length ℓ. The two masses both have an initial velocity v and the spring between them remains at its relaxed length. These masses then travel towards an identical setup (consisting of two masses connected by a spring) on a frictionless, horizontal table. If these four masses are aligned and undergo perfectly elastic, head-on collisions:
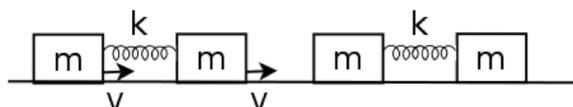

1. Determine the equations of motion of the masses after the first collision and before the second collision.

2. Determine the elapsed time between the first and second collisions, and show that there will be only a total of two collisions.

Problem 2.11 (Ricardo)**.** A particle moves on a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The cone has a half-angle of α. Let the particle move in a circle of radius r_0. (i) Find the frequency ω of the circular motion. (ii) Let the particle be perturbed slightly from this motion. Find the frequency Ω of oscillations about the radius r_0.

Problem 2.12 (Morin)**.** Two particles of mass m are constrained to move along two horizontal frictionless rails that make an angle 2θ with respect to each other. They are connected by a spring with spring constant k, whose relaxed length is shown below. What is the frequency of oscillations for the motion where the spring remains parallel to the position shown?
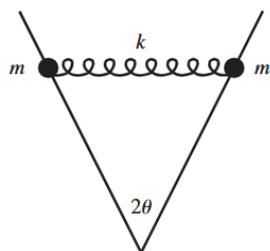
3   Advanced Problems

Problem 3.1. A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

1. Geometrically, what kind of curve is the resulting trajectory?

2. Find the value of v so that the projectile hits the ground traveling straight downward.

Problem 3.2 (SPOT TST 2017)**.** A point object of mass m slides on a frictionless parabolic curve in a uniform gravitational field g. The curve shown below has equation $y = \frac{x^2}{L}$ where L is a constant.
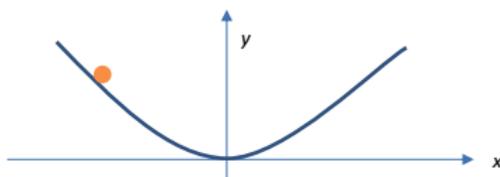

1. (a) Derive an expression for the total energy $E_{\text{parabolic}}$ of the point object, as a function only of x and $\dot{x}$, where $\dot{x} = \frac{dx}{dt}$.

    (b) Show that the motion of the point object is only *approximately* simple harmonic, and find the period T_{approx}.

2. Instead of the frictionless parabolic curve, we wish to find the frictionless tautochrone[1] curve, on which a point object will oscillate with the same period regardless of where it is released from. Let this common period be T_0.

    (a) Define the arc length s as the distance along the curve from the origin. Derive an expression for the total energy $E_{\text{tautochrone}}$ of the point object, written as a function only of s and $\dot{s}$.

    (b) Deduce the parametric equations defining the tautochrone curve, i.e. expressions $x(\theta)$, $y(\theta)$ where the angle θ is related to the slope of the curve by

$$\tan\theta = \frac{dy}{dx}.$$

Problem 3.3. A heavy bob is often used to stabilize buildings in the event of earthquakes. Let us consider a related problem. Two masses m_1 and m_2 are stationary on a horizontal, frictionless plane and are connected by a spring of spring constant k. Suppose a force

$$F(t) = f\cos\omega t$$

is exerted on m_1, in the direction of the line joining the two masses. Determine the value of k for which the particular solution to the equation of motion of m_1 yields an oscillation of *zero amplitude*. There is no damping.

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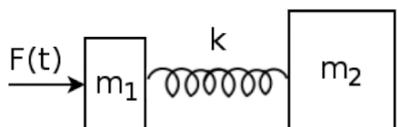
[1]On a tautochrone curve, a point object will oscillate with the same period regardless of its release point (i.e. period independent of amplitude, just like a simple harmonic oscillator).

Problem 3.4 (Ricardo)**.** The curve $y = b\left(\frac{x}{a}\right)^{\lambda}$ is rotated around the y-axis with constant frequency ω, and $a, b > 0$. A bead moves without friction along the curve. Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist?